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Long-Time Behavior for the 1-D Stochastic Ising Model with Unbounded Random Couplings

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We consider the ferromagnetic Ising model with Glauber spin flip dynamics in one dimension. The external magnetic field vanishes and the couplings are i.i.d. random variables. If their distribution has compact support, the disorder averaged spin auto-correlation function has an exponential decay in time. We prove that, if the couplings are unbounded, the decay switches to either a power law or a stretched exponential, in general.

KEY WORDS: Spin-flip dynamics; disordered energy; stretched exponential decay.

1. INTRODUCTION AND MAIN RESULTS

In one dimension the Ising model with spin flip dynamics has exponentially fast mixing in time, as is reflected by the fact that the self-adjoint generator of the stochastic dynamics has a spectral gap, see ref. 1 for example. One might wonder what happens to the exponential decay when the couplings are disordered. If the couplings are uniformly bounded, it is proved in refs. 2 and 3 that the generator still has a spectral gap. Thus the case of interest is when the couplings are unbounded. It is easy to see that then the spectral gap vanishes with probability one. The goal of our paper is to estimate how the missing spectral gap is reflected in the decay of the disorder averaged spin-spin correlation. In particular we will have to identify those realizations of the couplings which are responsible for a slow decay.

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Results on the relaxation to equilibrium of Glauber-type dynamics for disordered systems have been discussed before by many authors. One-dimensional stochastic spin models with unbounded random couplings were considered in refs. 3, 4, and 6, where stretched exponential relaxation under typical couplings was observed. Here we study the average behavior of the dynamics. Using a new approach, based on the oscillation theorem, we find an explicit relation between a probability distribution of the couplings and convergence properties of the disorder averaged spin auto-correlation function.

The model under the study is the one-dimensional Ising model with formal Hamiltonian

$$H(\sigma, \omega) = -\sum_{x \in \mathbb{Z}} \omega_x \sigma_{x-1} \sigma_x. \tag{1}$$

Here $\sigma_x = \pm 1$ are the Ising spins, $\sigma \in \Omega = \{1, -1\}^{\mathbb{Z}}$, and ω_x are the couplings. We assume that $\{\omega_x, x \in \mathbb{Z}\}$ are i.i.d. random variables with a common probability distribution P. The model is assumed to be ferromagnetic, $\omega_x \geqslant 0$, i.e., P is supported in $\mathbb{R}_+$. The family of random variables $\omega = \{\omega_x, x \in \mathbb{Z}\}$ is an ergodic random field on $\mathbb{Z}$ with the space of realizations $\mathbb{R}_+^{\mathbb{Z}}$ and the probability distribution $\mathbf{P} = P^{\mathbb{Z}}$. It is known that for every bounded realization of the random field ω and that for $\mathbf{P}$-a.e. unbounded ω the random spin system with Hamiltonian (1) has a unique limit Gibbs measure ν_ω for arbitrary inverse temperature β.[2, 3, 5, 6] To simplify our notation we include β into the definition of the coupling ω_x.

For a fixed realization of the couplings the Ising spin configuration σ evolves in time through spin flips as specified by the flip rates

$$c(x, \sigma, \omega) = \frac{1}{1 + e^{-\Delta_x(\sigma, \omega)}},$$

$$\Delta_x(\sigma, \omega) = H(\sigma^{(x)}, \omega) - H(\sigma, \omega), \qquad \sigma^{(x)} \in \Omega, \quad \sigma_y^{(x)} = \begin{cases} \sigma_y, & y \neq x, \\ -\sigma_y, & y = x. \end{cases}$$

Let us notice that our choice of the flip rates implies a Jacobi structure of the generator of the Glauber dynamics in the one-spin sector, see (8) below. In the general case the representation (8) doesn't hold.

Thus in a short time interval dt the spin configuration σ changes to the spin configuration $\sigma^{(x)}$ with probability $c(x, \sigma, \omega)\, dt$ and remains unchanged with probability $1 - \sum_x c(x, \sigma, \omega)\, dt$. It is proved in ref. 7 that this rule defines a Markov process, denoted here by

$$\sigma^\omega(t) = \{\sigma_x^\omega(t), x \in \mathbb{Z}, t \geqslant 0\},$$

with state space Ω. We assume that the $t=0$ distribution of $\sigma^\omega(t)$ is the Gibbs measure ν_ω. Then $\sigma^\omega(t)$ is stationary in time and reversible. The corresponding stochastic semigroup T_t is self-adjoint on the Hilbert space $\mathscr{H}_\omega = \mathscr{L}_2(\Omega, d\nu_\omega)$. T_t is generated by

$$(L(\omega) f)(\sigma) = \sum_{x \in \mathbb{Z}} c(x, \sigma, \omega)(f(\sigma^{(x)}) - f(\sigma)), \qquad f \in \mathscr{D} \subset \mathscr{H}_\omega, \tag{2}$$

as acting on cylindrical functions $\mathscr{D} \subset \mathscr{H}_\omega$. The operator $L(\omega)$ can be extended in $\mathscr{H}_\omega$ to a self-adjoint (unbounded) operator for **P**-a.e. ω[7, 3] and will be denoted by the same symbol. Let

$$\lambda_0 = \sup\{(L\psi, \psi), \|\psi\| = 1, (\psi, 1) = 0\}$$

denote the upper edge of the spectrum of the operator $L(\omega)$ in the subspace orthogonal to $\{1\}$. λ_0 is constant almost surely.

The goal of our paper is to obtain the long-time behavior for the disorder averaged time-autocorrelation function

$$S(t) = \langle \langle \sigma_0^\omega(t), \sigma_0(0) \rangle_{\mathscr{P}(\omega)} \rangle, \qquad t \to \infty. \tag{3}$$

Here $\langle \cdot \rangle_{\mathscr{P}(\omega)}$ is the average over the process $\sigma^\omega(t)$ under a fixed realization ω, and $\langle \cdot \rangle$ is the average over the distribution **P** of random couplings. We are interested in the case of unbounded couplings when

$$P(\omega_x > K) > 0 \qquad \text{for any} \quad K > 0.$$

In this case with probability one the operator L has no spectral gap, see for example ref. 3, which implies $\lambda_0 = 0$.

As our main result we state

Theorem 1. Let $P(\omega_x > K) > 0$ for every $K > 0$, and

$$1 < \langle (\cosh \omega_x)^4 \rangle < \infty. \tag{4}$$

Then for large enough t the following estimate holds

$$C_2 \left(\frac{t e^{-G_2(t)}}{\sqrt{-g_2''(\mu_2(t))}} \right)^2 \leqslant S(t) \leqslant C_1 \left(\frac{t e^{-G_1(t)}}{\sqrt{-g_1''(\mu_1(t))}} \right)^{\frac{1}{2}} \tag{5}$$

with positive constants C_1, C_2 independent on t. Here

$$g_1(\mu) = \ln P\left(\omega_x > \frac{1}{4} \ln \frac{1}{\mu} \right), \qquad g_2(\mu) = 2 \ln P\left(\omega_x > \frac{1}{2} \ln \frac{1}{c\ \mu} \right),$$

with $\mu \in (0, 1)$ for suitable constant c, $0 < c < 1$. G_j is the Legendre transform of g_j,

$$G_j(t) = \min_{\mu \in (0,1)} (t\mu - g_j(\mu)), \qquad t > 0, \quad j = 1, 2,$$

where the minimum is taken at $\mu_j(t)$.

Examples. (i) If $P(\omega_x > u) \sim e^{-ku}$ for $u \to \infty$ with $k > 4$, then one has

$$g_1(\mu, t) = \frac{k}{4} \ln \mu, \qquad \mu \in (0, 1), \qquad \mu_1(t) = \frac{k}{4t},$$

$$g_2(\mu, t) = k \ln c\mu, \qquad \mu \in (0, 1), \qquad \mu_2(t) = \frac{k}{t},$$

and

$$C_2(1+t)^{-2k} \leqslant S(t) \leqslant C_1(1+t)^{-\frac{k}{8}}$$

with constants C_1, C_2 independent on t.

(ii) If $P(\omega_x > u) \sim e^{-u^\alpha}$ for $u \to \infty$ with $\alpha > 1$, then one has

$$g_1(\mu, t) = -\left(\frac{1}{4}\right)^\alpha \left(\ln \frac{1}{\mu}\right)^\alpha, \qquad \mu \in (0, 1),$$

$$\mu_1(t) = \alpha \left(\frac{1}{4}\right)^\alpha \frac{(\ln t)^{\alpha-1}}{t} (1 + o(1)), \qquad t \to \infty,$$

$$g_2(\mu, t) = -2\left(\frac{1}{2}\right)^\alpha \left(\ln \frac{1}{c\mu}\right)^\alpha, \qquad \mu \in (0, 1),$$

$$\mu_2(t) = 2\alpha \left(\frac{1}{2}\right)^\alpha \frac{(\ln t)^{\alpha-1}}{t} (1 + o(1)), \qquad t \to \infty,$$

and

$$C_2 e^{-4\left(\frac{1}{2}\right)^\alpha (\ln t)^\alpha (1+o(1))} \leqslant S(t) \leqslant C_1 e^{-\frac{1}{2}\left(\frac{1}{4}\right)^\alpha (\ln t)^\alpha (1+o(1))}$$

with some positive constants C_1, C_2.

Our analysis estimates the integrated density of states of the generator L in the one-spin sector near zero. Using techniques from the oscillation theorem (see refs. 8 and 11, for example) we establish a relation between realizations of the random couplings and the integrated density of states. This approach is exploited already in refs. 9 and 10 in the case of bounded ω. There the sub-leading correction to the exponential decay of $S(t)$ is determined through an analysis of the asymptotics of the density of states. For unbounded couplings, however, a new mechanism appears resulting in a novel behavior for the spectral characteristics of the generator. For bounded couplings the spectrum near the upper edge comes from low probability, atypical random couplings, for which there are long stretches close to the maximum. They result in a Lifschitz tail in the integrated density of states. This type of spectrum boundary is called fluctuation boundary. In contrast, as follows from the arguments given below, for unbounded couplings the main contribution to the spectrum close to zero comes from rapid oscillations of the couplings over short intervals. This behavior of the spectral characteristics of the generator determines the leading decay of the disorder averaged auto-correlation function. In particular, this implies that the sub-leading decay for bounded couplings is unrelated to the leading decay for unbounded couplings.

2. REDUCING SUBSPACE, PROOF OF THEOREM 1

The auto-correlation (3) can be rewritten as follows

$$S(t) = \langle (e^{tL(\omega)}\sigma_0, \sigma_0) \rangle = \langle (e^{tL_1(\omega)}\sigma_0, \sigma_0) \rangle. \tag{6}$$

We must explain the meaning of the operator $L_1 = L_1(\omega)$. Since the work of R. Glauber[12] it is known that the linear span of "one-point configurations" $\{\sigma_x, x \in \mathbb{Z}\}$ forms an invariant subspace for the generator $L(\omega)$ of (2). Moreover, the same invariant subspace $\mathscr{H}_1(\omega) \subset \mathscr{H}_\omega$ is spanned by the functions

$$v_x(\sigma, \omega) = \cosh \omega_x \cdot \sigma_x - \sinh \omega_x \cdot \sigma_{x-1}, \qquad x \in \mathbb{Z}, \tag{7}$$

see ref. 2. The functions (7) form the orthonormal basis in $\mathscr{H}_1(\omega)$. We denote by $L_1(\omega)$ the restriction of the generator $L(\omega)$ to the invariant subspace $\mathscr{H}_1(\omega)$.

The operator L_1 has the following symmetric representation in the basis $\{v_x, x \in \mathbb{Z}\}$

$$L_1 v_x = A_{x,x-1} v_{x-1} + A_{x,x} v_x + A_{x,x+1} v_{x+1}, \tag{8}$$

with

$$A_{x,x-1} = A_{x-1,x} = \frac{a_x \sqrt{(1-a_x^2)(1-a_{x-1}^2)}}{(1-a_x^2 a_{x-1}^2)}, \qquad a_x = \tanh \omega_x > 0,$$

$$A_{x,x} = -1 - \frac{a_x^2(1-a_{x-1}^2)}{(1-a_x^2 a_{x-1}^2)} + \frac{a_{x+1}^2(1-a_x^2)}{(1-a_x^2 a_{x+1}^2)}.$$

We consider new random variables

$$C_x = \frac{a_x^2(1-a_{x-1}^2)}{(1-a_x^2 a_{x-1}^2)}, \qquad C_x \in (0, 1). \tag{9}$$

Then

$$A_{x,x-1} = \sqrt{C_x(1-C_x)}\,, \qquad A_{x,x} = -1 - C_x + C_{x+1}.$$

Using the representation (8) one can introduce the integrated density of states $N(L_1, d\lambda)$ for the random operator L_1 by the truncated operators $L_1^{(r)}$, $r \in \mathbb{N}$, defined on a finite-dimensional space of functions V_r of the form

$$V_r = \left\{ f^{(r)}(\sigma) = \sum_{x=-r}^{r} f_x v_x \right\} \subset \mathscr{H}_1(\omega).$$

Let P_{V_r} be the projection on V_r. Then the truncated operator $L_1^{(r)} = P_{V_r} L_1 P_{V_r}$ is given by the same formula as (8) when $x = -r,..., r$. We denote by $0 \geqslant \lambda_1^{(r)} \geqslant \lambda_2^{(r)} \geqslant \cdots \geqslant \lambda_{2r+1}^{(r)}$ the eigenvalues of the truncated operator $L_1^{(r)}$ in decreasing order and by $k(L_1^{(r)}, \lambda)$ the number of eigenvalues of $L_1^{(r)}$ exceeding $\lambda \in \mathbb{R}$. Then from results in ref. 11 it follows that there exists a non-random positive measure $N(L_1, d\lambda)$ on $\mathbb{R}$, such that with probability one

$$\lim_{r \to \infty} \frac{1}{2r+1} k(L_1^{(r)}, \lambda) = N(L_1, \lambda)$$

in the sense of weak convergence of measures, where

$$N(L_1, \lambda) = N(L_1, (\lambda, +\infty)).$$

In addition

$$N(L_1, \lambda) = \langle (E_{L_1}(\lambda, +\infty)\, v_0, v_0) \rangle, \tag{10}$$

where $\{E_{L_1}(d\lambda)\}$ is the spectral resolution of the operator L_1. The representations (8) to (9) imply (see Lemma 1 below) that the measure $N(L_1, d\lambda)$ is concentrated on $\mathbb{R}_-$, so that $N(L_1, \lambda) = N(L_1, (\lambda, 0))$ for negative λ.

Main Lemma. Let $\lambda < 0$ with $|\lambda|$ sufficiently small. Then

$$N(L_1, \lambda) \geqslant C_2 \left[P\left(\omega_x > \frac{1}{2} \ln \frac{1}{c\,|\lambda|} \right) \right]^2 = C_2 e^{g_2(|\lambda|)}, \tag{11}$$

$$N(L_1, \lambda) \leqslant C_1 P\left(\omega_x > \frac{1}{4} \ln \frac{1}{|\lambda|} \right) = C_1 e^{g_1(|\lambda|)} \tag{12}$$

with positive constants C_j, $j = 1, 2$ and a constant c, $0 < c < 1$.

The proof of the main lemma will be given in Sections 3 and 4 below. We first derive the asymptotic formula (5) based on the estimates (11), (12).

Proof of Theorem 1. (1) *The upper bound.* Since, see refs. 2 and 9,

$$\sigma_x = \sum_{y \leqslant x} D_{x,y}(\omega)\, v_y, \tag{13}$$

with

$$\begin{aligned} D_{x,y}(\omega) &= (1 - \tanh^2 \omega_y)^{1/2} \tanh \omega_{y+1} \cdots \tanh \omega_x, \qquad y < x, \\ D_{x,x}(\omega) &= (1 - \tanh^2 \omega_x)^{1/2}, \end{aligned} \tag{14}$$

we have

$$\begin{aligned} \langle (e^{tL_1}\sigma_0, \sigma_0) \rangle &= \sum_{x \leqslant 0} \sum_{y \leqslant 0} \langle D_{x,0} D_{y,0} (e^{tL_1} v_x, v_y) \rangle \\ &\leqslant \sum_{x \leqslant 0} \sum_{y \leqslant 0} \langle D_{x,0}^2 D_{y,0}^2 \rangle^{1/2} \langle (e^{tL_1} v_x, v_y)^2 \rangle^{1/2} \\ &\leqslant \left(\sum_{x \leqslant 0} \langle D_{x,0}^4 \rangle^{1/4} \right)^2 \langle (e^{tL_1} v_x, v_y)^2 \rangle^{1/2}. \end{aligned} \tag{15}$$

The representation (14) together with the condition (4) on the distribution of the random variables ω_x imply that for any $x < 0$

$$\langle D_{x,0}^4 \rangle = \langle (1 - \tanh^2 \omega_x)^2 \rangle \langle \tanh^4 \omega_{x+1} \rangle \cdots \langle \tanh^4 \omega_0 \rangle \leqslant \kappa^{|x|},$$

with some $0 < \kappa < 1$, so that

$$\sum_{x \leqslant 0} \langle D_{x,0}^4 \rangle^{1/4} \leqslant C = C(\kappa). \tag{16}$$

Furthermore, for every x, y

$$(e^{tL_1}v_x, v_y)^2 \leqslant (e^{tL_1}v_x, v_x)(e^{tL_1}v_y, v_y) \leqslant (e^{tL_1}v_x, v_x). \tag{17}$$

Finally from (6), (15)–(17), (10), and (12) we conclude that for large t

$$\begin{aligned} S(t) &= \langle (e^{tL_1}\sigma_0, \sigma_0)\rangle \leqslant C^2 \langle (e^{tL_1}v_0, v_0)\rangle^{1/2} \\ &= C^2 \left(\left\langle \int_R e^{t\lambda} (E_{L_1}(d\lambda)\, v_0, v_0) \right\rangle \right)^{1/2} \\ &= C^2 \left(\int_{-\infty}^0 e^{t\lambda} N(L_1, d\lambda) \right)^{1/2} \\ &\leqslant \tilde{C}_1 \left(t \int_0^\infty e^{-t\mu + g_1(\mu)}\, d\mu \right)^{1/2} \leqslant C_1 \left(\frac{t e^{-G_1(t)}}{\sqrt{-g_1''(\mu_1(t))}} \right)^{1/2}, \end{aligned}$$

where g_1, G_1, and $\mu_1(t)$ are defined in Theorem 1.

(2) *The lower bound.* By (7) and (4) we obtain in analogy with the above reasoning that

$$\begin{aligned} &\langle (e^{tL_1}v_0, v_0)\rangle \\ &\quad = \langle (e^{tL_1}(\sigma_0 \cosh \omega_0 - \sigma_{-1} \sinh \omega_0), \sigma_0 \cosh \omega_0 - \sigma_{-1} \sinh \omega_0)\rangle \\ &\quad \leqslant \langle \cosh^2 \omega_0 (e^{tL_1}\sigma_0, \sigma_0)\rangle + \langle \sinh^2 \omega_0 (e^{tL_1}\sigma_{-1}, \sigma_{-1})\rangle \\ &\qquad + 2\langle \cosh \omega_0 \sinh \omega_0\, |(e^{tL_1}\sigma_0, \sigma_{-1})|\rangle \leqslant 2\langle \cosh^4 \omega_0\rangle^{1/2} \langle (e^{tL_1}\sigma_0, \sigma_0)^2\rangle^{1/2} \\ &\qquad + 2\langle \cosh^2 \omega_0 \sinh^2 \omega_0\rangle^{1/2} \langle (e^{tL_1}\sigma_0, \sigma_{-1})^2\rangle^{1/2} \leqslant k_1 \langle (e^{tL_1}\sigma_0, \sigma_0)\rangle^{1/2} \end{aligned} \tag{18}$$

with some constant k_1, where we used the estimate

$$(e^{tL_1}\sigma_0, \sigma_{-1})^2 \leqslant (e^{tL_1}\sigma_0, \sigma_0)(e^{tL_1}\sigma_{-1}, \sigma_{-1}) \leqslant (e^{tL_1}\sigma_0, \sigma_0).$$

Now from (6), (18), (10), and (11) we derive for large t the lower bound on $S(t)$ as

$$\begin{aligned} S(t) &= \langle (e^{tL_1}\sigma_0, \sigma_0)\rangle \geqslant k \langle (e^{tL_1}v_0, v_0)\rangle^2 \\ &= k \left(\int_{-\infty}^0 e^{t\lambda} N(L_1, d\lambda) \right)^2 \geqslant \tilde{C}_2 \left(t \int_0^\infty e^{-t\mu + g_2(\mu)}\, d\mu \right)^2 \\ &\geqslant C_2 \left(\frac{t e^{-G_2(t)}}{\sqrt{-g_2''(\mu_2(t))}} \right)^2, \end{aligned}$$

where g_2, G_2, and $\mu_2(t)$ are defined in Theorem 1. This completes the proof of the theorem. ▮

3. THE ESTIMATE OF $N(L_1, \lambda)$ FROM BELOW

Let us fix the configuration $\omega = \{\omega_x, x \in \mathbb{Z}\}$. The truncated operator $L_1^{(r)}(\omega)$ defined above by (8) is given by a Jacobi symmetric matrix of the order $2r+1$ with positive entries $A_{x,x-1}$, $x = -r+1,..., r$. Consequently, for any r the operator $L_1^{(r)}$ has only real eigenvalues and we can exploit the technique of the oscillation theorem in the spectral analysis for $L_1^{(r)}$.

Lemma 1. For every r and $f \in V_r$ one has

$$0 \leqslant (-L_1^{(r)} f, f) \leqslant 2 \, \|f\|^2.$$

Proof. The proof easily follows from the obvious inequalities

$$2\sqrt{C_y(1-C_y)}\, f_y f_{y-1} \leqslant (1-C_y)\, f_{y-1}^2 + C_y f_y^2,$$

$$2\sqrt{C_y(1-C_y)}\, f_y f_{y-1} \geqslant -(1-C_y)\, f_y^2 - C_y f_{y-1}^2. \quad ▮$$

Lemma 1 implies that the operators $L_1^{(r)}$ have only negative real eigenvalues $\lambda_j^{(r)}(\omega)$, $j = 1,..., 2r+1$. First we evaluate the function $k(L_1^{(r)}, \lambda)$ from below for $\lambda < 0$.

Definition. We call a bond $\{x, x+1\}$ regular, if the random variables C_x and C_{x+1}, defined by (9), satisfy the condition

$$1 + C_x - C_{x+1} < |\lambda|. \tag{19}$$

Then the following estimate holds.

Lemma 2. For given $\lambda < 0$

$$k(L_1^{(r)}, \lambda) \geqslant \mathcal{R}_r(\lambda), \tag{20}$$

where $\mathcal{R}_r(\lambda)$ is the number of regular pairs, arranged on the interval $[-r, r]$ without overlapping.

Proof. To calculate the number of eigenvalues of $L_1^{(r)}$ exceeding $\lambda < 0$ we will exploit the oscillation theorem to the operator $-L_1^{(r)}$ and estimate the number $\tilde{k}(-L_1^{(r)}, |\lambda|)$ of eigenvalues of $-L_1^{(r)}$ not exceeding $|\lambda|$:

$k(L_1^{(r)}, \lambda) = \tilde{k}(-L_1^{(r)}, |\lambda|)$. Let $\{f_x(\lambda)\}$ is an eigenfunction of $-L_1^{(r)}$ corresponding to an eigenvalue $|\lambda|$. We define the standard phase $\varphi_x(\omega)$ by

$$\operatorname{ctg} \varphi_{x+1}(\omega) = \operatorname{ctg} \varphi_{x+1} = \frac{f_{x+1}(\lambda)}{f_x(\lambda)}, \qquad x = -r, \ldots, r-1.$$

Then

$$\operatorname{ctg} \varphi_{x+1} = \frac{1 + C_x - C_{x+1} - |\lambda|}{\sqrt{C_{x+1}(1 - C_{x+1})}} - \frac{\sqrt{C_x(1 - C_x)}}{\sqrt{C_{x+1}(1 - C_{x+1})}} \cdot \frac{1}{\operatorname{ctg} \varphi_x}. \tag{21}$$

By the oscillation theorem $\tilde{k}(-L_1^{(r)}, |\lambda|) = m_r(J(\lambda)) + 1$, where $J(\lambda) \leqslant |\lambda|$ is the maximal eigenvalue of $-L_1^{(r)}$ not exceeding $|\lambda|$, and $m_r(\omega, J(\lambda))$ is the number of sign changes in the sequence of coordinates $\{f_x(J(\lambda))\}$, $x = -r, \ldots, r$ of the corresponding eigenfunction. Thus $m_r(J(\lambda))$ equals the number of sites $x \in [-r, r]$ with $\operatorname{ctg} \varphi_x < 0$,

$$m_r(J(\lambda)) = \#\{x \in [-r, r] : \operatorname{ctg} \varphi_x < 0\}.$$

Let us consider a regular bond $\{x, x+1\}$. If $\operatorname{ctg} \varphi_x < 0$, then we already have a contribution to $m_r(J(\lambda))$ from that bond. If $\operatorname{ctg} \varphi_x > 0$, then (21) and (19) imply that $\operatorname{ctg} \varphi_{x+1} < 0$. So in any case we have a contribution to $m_r(J(\lambda))$ from each regular bond. Lemma 2 is proved. ∎

Finally by averaging the inequality (20) over realizations ω and taking the limit $r \to \infty$ we have for $\lambda < 0$,

$$N(L_1, \lambda) = \lim_{r \to \infty} \frac{\langle k(L_1^{(r)}, \lambda) \rangle}{2r+1} \geqslant b\mathbf{P}(1 + C_0 - C_1 < |\lambda|) \tag{22}$$

with some constant b. We estimate the probability $\mathbf{P}(1 + C_0 - C_1 < |\lambda|)$ under sufficiently small $|\lambda|$ in terms of the distribution P of ω_x.

Lemma 3. For all sufficiently small $|\lambda|$

$$\mathbf{P}(1 + C_0 - C_1 < |\lambda|) \geqslant p_0 \left[P\left(\omega_x > \frac{1}{2} \ln \frac{1}{c\,|\lambda|} \right) \right]^2 \tag{23}$$

with constants $0 < p_0 < 1$ and $0 < c < 1$.

Proof. Let us fix some constant h, $0 < h < 1$, and we denote by

$$p_0 = P(0 < \tanh \omega_x < h), \qquad 0 < p_0 < 1.$$

Then using the representation (9) for C_x we have for small enough $|\lambda|$

$$\begin{aligned}\mathbf{P}(C_0+1-C_1<|\lambda|) &\geqslant \mathbf{P}(C_0<|\lambda|/2;\ 1-C_1<|\lambda|/2)\\ &= \mathbf{P}\left(\frac{a_0^2(1-a_{-1}^2)}{1-a_0^2a_{-1}^2}<|\lambda|/2;\ \frac{1-a_1^2}{1-a_0^2a_1^2}<|\lambda|/2\right)\\ &\geqslant \mathbf{P}(a_{-1}>1-\tilde{c}_0\,|\lambda|;\ 0<a_0<h;\ a_1>1-\tilde{c}_0\,|\lambda|)\\ &= p_0[P(a_x>1-\tilde{c}_0|\lambda|)]^2 = p_0\left[P\left(\omega_x>\frac{1}{2}\ln\frac{1}{c\,|\lambda|}\right)\right]^2\end{aligned}$$

with some $\tilde{c}_0$ and $0<c<1$. ∎

The estimate (11) on $N(L_1, \lambda)$ from below follows from (22) and (23).

4. THE ESTIMATE OF $N(L_1, \lambda)$ FROM ABOVE

For given $\lambda<0$ we denote by $\gamma(\lambda)=\frac{1}{4}\ln\frac{1}{|\lambda|}$. Then for any configuration $\omega=\{\omega_x,\ x\in\mathbb{Z}\}$ we consider a decomposition of $\mathbb{Z}$ into two sets,

$$\mathbb{Z}=A_{\omega,\lambda}\cup B_{\omega,\lambda}$$

with

$$A_{\omega,\lambda}=\{x\in\mathbb{Z}:\omega_x>\gamma(\lambda)\},\qquad B_{\omega,\lambda}=\{x\in\mathbb{Z}:\omega_x\leqslant\gamma(\lambda)\}. \tag{24}$$

For any $r\in\mathbb{N}$ we denote by

$$\begin{gathered}B^0_{r,\omega,\lambda}=\{x\in[-r,r]:\max\{\omega_x,\omega_{x-1},\omega_{x+1}\}\leqslant\gamma(\lambda)\}\\ B^0_{r,\omega,\lambda}\subset B_{\omega,\lambda}\cap[-r,r],\end{gathered} \tag{25}$$

and by $W_{r,\lambda}\subset V_r$ the linear span of functions $\{v_x,\ x\in B^0_{r,\omega,\lambda}\}$. Then the operators

$$L_1^{(r)}=P_{V_r}L_1P_{V_r},\qquad L_1^{(W_{r,\lambda})}=P_{W_{r,\lambda}}L_1P_{W_{r,\lambda}} \tag{26}$$

are truncations of L_1 on subspaces V_r and $W_{r,\lambda}$ respectively. Since L_1 is a self-adjoint bounded operator, we have by the minimax principle

$$k(L_1^{(W_r)},\lambda)\geqslant k(L_1^{(r)},\lambda)-\#\{x\in[-r,r],\ x\notin B^0_{r,\omega,\lambda}\}, \tag{27}$$

where, as above, $k(A,\lambda)$ denotes the number of eigenvalues of the operator A exceeding λ.

Lemma 4. For any sufficiently small $|\lambda|$, $\lambda < 0$, and for every ω we have

$$k(L_1^{(W_r,\lambda)}(\omega), \lambda) = 0, \tag{28}$$

where $L_1^{(W_r,\lambda)}(\omega)$ is defined in (26).

Proof. We consider the bounded configuration

$$\tilde{\omega} = \{\tilde{\omega}_x \leqslant \gamma(\lambda),\ x \in \mathbb{Z}\},$$

coinciding with the configuration ω on $B^0_{r,\omega,\lambda}$,

$$\tilde{\omega}_x = \omega_x,\ x \in B^0_{r,\omega,\lambda}.$$

Let $L_1^\gamma(\tilde{\omega})$ be an operator in $\mathscr{H}_1(\tilde{\omega})$ given by (8), (9) and corresponding to the configuration $\tilde{\omega}$. Our constructions (24)–(25) imply that the operator $L_1^{(W_r,\lambda)}(\omega)$ is the same as the truncation of the operator $L_1^\gamma(\tilde{\omega})$ on the same subspace $W_{r,\lambda}$. As follows from results of ref. 2 in the case of bounded couplings, under the assumption $\tilde{\omega}_x < \gamma(\lambda)$ the upper spectrum edge of the operator $L_1^\gamma(\tilde{\omega})$ equals to

$$\lambda_0 = -1 + \tanh 2\gamma(\lambda)$$

for a.e.-configuration $\tilde{\omega}$, so that $\lambda_0 < \frac{3}{2}\lambda$ for small enough $\lambda < 0$. This estimate is valid also for any truncation of the operator $L_1^\gamma(\tilde{\omega})$. Thus no eigenvalue of the operator $L_1^{\gamma,(r)}(\tilde{\omega})$ or $L_1^{(W_r,\lambda)}(\omega)$ can be greater than λ. ∎

By (27) and (28) we have the following estimate

$$\begin{aligned} k(L_1^{(r)}(\omega), \lambda) &\leqslant \#\{x \in [-r, r],\ x \notin B^0_{r,\omega,\lambda}\} \\ &\leqslant C\#\{x \in [-r, r],\ x \in A_{\omega,\lambda}\} \end{aligned} \tag{29}$$

with some constant C. Applying, as before, the ergodic theorem to the inequality (29), we obtain the estimate (12) on $N(L_1, \lambda)$ from above,

$$N(L_1, \lambda) = \lim_{r \to \infty} \frac{\langle k_r(\omega, \lambda) \rangle}{2r+1} \leqslant C_1 P\left(\omega_x > \frac{1}{4} \ln \frac{1}{|\lambda|}\right).$$

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